3. Vatazhin, A. B. and Grabovskii, V. I., The spreading of singly ionized jets in hydrodynamic streams. PMM Vol. 37, № 1, 1973.
4. Grabovskii, V. I., Plane electrohydrodynamic flow with reverse current. PMM Vol. 37, № 5, 1973.
5. Vereshchagin, V. P., Levitov, V. I., Merzabekian, G. Z. and Pashin, M. M., Fundamentals of Electrogasdynamics of Disperse Systems. "Energiia", Moscow, 1974.

Translated by J. J. D.

UDC 533.6.011

ON THE PROPAGATION OF SMALL PERTURBATIONS IN A SONIC STREAM AND IN A QUIESCENT GAS

PMM Vol. 40, № 1, 1976, pp. 74-78

P. A. VEL'MISOV
(Ul'ianovsk)

(Received June 2, 1975)

Approximate equations are derived for small unsteady perturbations of a constant sonic stream and of quiescent gas. These equations, unlike the equation used for defining unstable transonic flows of gas, provide a correct definition of perturbation propagation from a point source in all directions [1].

1. Let us consider potential flows of perfect gas. Such flows are defined by the equation

$$(\Phi_t + V^2)_t + \Phi_x^2\Phi_{xx} + \Phi_y^2\Phi_{yy} + \Phi_z^2\Phi_{zz} + 2\Phi_x\Phi_y\Phi_{xy} + 2\Phi_x\Phi_z\Phi_{xz} + 2\Phi_y\Phi_z\Phi_{yz} = a^2(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}) \quad (1.1)$$

$$a^2 = \rho^{\varkappa-1} = P^{(\varkappa-1)/\varkappa} = \frac{\varkappa+1}{2} - (\varkappa - 1)\left(\Phi_t + \frac{1}{2}V^2\right)$$

$$V = (\Phi_x^2 + \Phi_y^2 + \Phi_z^2)^{1/2}$$

where Φ, x, y, z, t, V, a, P and ρ are, respectively, the dimensionless velocity potential, Cartesian coordinates, time, velocity of gas, speed of sound, pressure and density (related, respectively, to $a_*^2 t_0$, $a_* t_0$, t_0, a_*, P_* and ρ_*, where the asterisk denotes parameters of the sonic stream $u = \Phi_x = a_*$ and $\Phi_y = 0$).

Let us consider transonic flows of gas, for which it is possible to use the linear theory

$$\Phi = x + \gamma\Phi_1 + \gamma^2\Phi_2 + \ldots, \quad \Phi_{1tt} + 2\Phi_{1xt} = \Phi_{1yy} + \Phi_{1zz} \quad (1.2)$$

$$\gamma \ll 1$$

However the linear theory has some shortcomings. The linear expansion (1.2) contains various irregularity regions for which the order of the second term $\gamma^2\Phi_2$ is the same as of the first $\gamma\Phi_1$.

As an example, we present two such expansions for one-dimensional flows

$$\Phi = x + \gamma\beta_1(\nu) + \gamma^2\left[-\frac{\varkappa+1}{8}\beta_1'^2(\nu)x + \beta_2(\nu)\right] + \ldots, \quad (1.3)$$

$$\nu = 2t - x$$

$$\Phi = x + \gamma\alpha_1(x) + \gamma^2\left[-\frac{\varkappa+1}{8}\alpha_1'^2(x)\nu + \alpha_2(x)\right] + \ldots \quad (1.4)$$

These expansions determine the flow in a channel induced by a piston whose motion in a supersonic stream is defined by $x = t + \gamma h_1(t) + \ldots$, as well as for small perturbations in a quiescent gas ([2], p. 247)

$$\Phi = \gamma\Phi_1 + \gamma^2\Phi_2 + \ldots, \quad \Phi_1(x,t) = \alpha\left(\sqrt{\frac{\varkappa+1}{2}}\,t + x\right) + \beta\left(\sqrt{\frac{\varkappa+1}{2}}\,t - x\right)$$

for which in (1.2) in solutions for Φ_2 and $\Phi_1 = \alpha_1(x) + \beta_1(\nu)$ only functions dependent on ν are retained in the case of (1.3) ($\Phi_1 = \beta_1$ is the departing wave), and on x in the case of (1.4) ($\Phi_1 = \alpha_1$ is the oncoming wave). Functions α_k and β_k are determined by conditions at the piston.

Expansions (1.3) and (1.4) become irregular for $\nu \sim 1$ and $x \sim 1/\gamma$, and for $x \sim 1$ and $\nu \sim 1/\gamma$, respectively. It can be readily shown that for considerable times $t \sim 1/\gamma$ in the neighborhood of shock waves, which in the first approximation coincide with the characteristics, $x = 2t + \gamma x_1(t)$ in the case of (1.3) (propagation downstream in a sonic flow) and $x = \gamma x_1(t) + \ldots$ in the case of (1.4) (propagation upstream).

Note that by substituting in (1.3) and (1.4) the variables ξ_1, ξ_2, $x = p\xi_1 + q\xi_2$, $\nu = l\xi_1 + m\xi_2$ with $ql \neq pm$, for the variables x and ν it is possible to rewrite expansions (1.3) and (1.4) in such form that their irregularity occurs for $\xi_k = s_1/\gamma$, $l\xi_1 + m\xi_2 = \theta_1$ and $\xi_k = s_2/\gamma$, $p\xi_1 + q\xi_2 = \theta_2$, respectively (in particular for considerable times $t \sim 1/\gamma$ and $\xi_1 \sim 1/\gamma$, $m = -q$, or for $\xi_2 \sim 1/\gamma$, $l = -p$). In the above formulas $\gamma \ll 1$; s_k and θ_k are variables of order unity, and ξ_k is one of the variables ξ_1 or ξ_2. This is taken into account in the selection and extension of variables in the course of equation derivation for two- and three-dimensional perturbations.

The anaysis of the flow in the irregularity regions of expansions (1.3) and (1.4) necessitates the introduction of the following expansions:

$$\Phi = x + \gamma\psi_1(s_k, \theta_k) + \gamma^2\psi_2(s_k, \theta_k) + \ldots \qquad (1.5)$$

The equation for ψ_1 and the general solution of these equations are readily obtained by substituting (1.5) into (1.1). A uniformly suitable solution is then obtained by joining ψ_1 with the linear solution $\Phi = \alpha_1(x) + \beta_1(\nu)$ by the method of merging asymptotic expansions [2, 3]. A typical form of equations for ψ_1 is provided, for instance, by the following equation obtained for $\xi_1 = x = \gamma^{-1}s$ and $\theta_1 = \nu = 2t - x$:

$$-\psi_{1s\nu} = (\varkappa + 1)\psi_{1\nu}\psi_{1\nu\nu}$$

2. Using the results obtained for one-dimensional flows, we obtain for two- and three-dimensional perturbations such nonlinear equations that, in the case of one-dimensional flow could be joined for $\Phi_1 = \Phi_1(x, t)$ with the solution of Eq. (1.2). One of such equations is evidently the equation which in the one-dimensional case consists of terms $\psi_{\xi\eta}$ and $\psi_\xi\psi_{\xi\xi}$ (taking into account the note on the form of variables for one-dimensional expansions, we introduce here $\xi = dx + bt$ and $\eta = kx + nt$) and contains besides these terms with derivatives with respect to y and z. In the general case this requirement yields the expansion (2.1)

$$\Phi = x + \delta\sum_{k=0}^{\infty}\psi_k(\xi^\circ, \eta^\circ, y^\circ, z^\circ)\left(\frac{\delta}{\varepsilon}\right)^k, \quad \xi = \varepsilon\xi^\circ, \quad \eta = \varepsilon^2\delta^{-1}\eta^\circ$$

$$y = \varepsilon^{2/3}\delta^{-1/2}y^\circ, \quad z = \varepsilon^{2/3}\delta^{-1/2}z^\circ, \quad \delta/\varepsilon \ll 1$$

where δ and ε are some constant positive parameters whose order of magnitude determines various regions of flow, and ξ°, η°, y° and z° are new variables of order unity. Substituting (2. 1) into (1. 1) and using the condition of nontriviality of the equation for ψ_0, we obtain that $b = -2d$ or $b = 0$. For ψ_0 we have the equation

$$2\,(bn + dn + bk)\psi_{\xi\eta} + (\varkappa + 1)d^2\,(b + d)\psi_\xi\psi_{\xi\xi} = \psi_{yy} + \psi_{zz} \qquad (2.2)$$

where the superscript $^\circ$ at variables is omitted and the notation $\psi_0 = \psi$ is used here and henceforth.

Specifying the equation of the sonic surface ($a^2 = V^2$) in the form

$$\xi = \varepsilon \sum_{k=0}^{\infty} \xi_k(\eta^\circ, y^\circ, z^\circ)\left(\frac{\delta}{\varepsilon}\right)^k, \quad \xi^\circ = \sum_{k=0}^{\infty} \xi_k(\eta^\circ, y^\circ, z^\circ) \qquad (2.3)$$

for the determination of function $\xi^\circ = \xi_0\,(\eta^\circ, y^\circ, z^\circ)$ in the first approximation we obtain the equation $\psi_\xi\,(\xi_0, \eta, y, z) = 0$.

Let us derive approximate conditions at the shock wave, which for the normal and tangent velocity components at transition through the wave front are of the form

$$(U - V_n)(U - V_n^*) = \frac{\varkappa - 1}{\varkappa + 1}(U - V_n^*)^2 + \frac{2}{\varkappa + 1}a^{*2}, \quad V_\tau = V_\tau^* \qquad (2.4)$$

where U is the shock wave velocity along the normal to the wave front, V_n and V_τ are the gas velocity components normal to the wave front, and an asterisk superscript denotes flow ahead of the shock wave. The speed of sound a^* is determined by the second of formulas (1. 1) with $\Phi = \Phi^*$. Defining the shock wave in the form (2. 3) and substituting (2. 1) and (2. 3) into (2. 4) in the first approximation for $\xi^\circ = \xi_0\,(\eta^\circ, y^\circ, z^\circ)$ we obtain

$$2\,(bn + dn + bk)\frac{\partial \xi_0}{\partial \eta} + \left(\frac{\partial \xi_0}{\partial y}\right)^2 + \left(\frac{\partial \xi_0}{\partial z}\right)^2 = \frac{d^2}{2}(b + d)(\varkappa + 1)\,(\psi_\xi + \psi_\xi^*), \quad \psi = \psi^* \qquad (2.5)$$

where the superscript $^\circ$ has been omitted.

Functions ψ and ψ^* and their derivatives in (2. 5) are taken for $\xi = \xi_0$. If $\psi^* \equiv \psi$, then (2. 5) is the equation of characteristics for (2. 2).

Let us consider the problem of selecting constants k and n ($\eta = kx + nt$). For this we, first, determine the shape of the perturbation front originating at point $x = y = 0$ for $t = 0$ in a uniform transonic stream $\psi^* = \omega\xi$. For simplicity we consider a plane flow. We substitute $\psi^* = \psi = \omega\xi$ into (2. 5) and obtain for the derived equation the solution

$$\xi = A\eta + By^2\eta^{-1}, \quad A = \frac{(\varkappa + 1)\,d^2\,(b + d)\,\omega}{2\,(bn + dn + bk)} \qquad (2.6)$$
$$2B = bn + dn + bk \neq 0$$

Passing to physical variables x, y, t we obtain (2. 6) in the form

$$(dk - rk^2)x^2 + (dn + bk - 2rkn)xt + (bn - rn^2)t^2 = By^2 \qquad (2.7)$$
$$r = \frac{\varepsilon}{\delta}A$$

Let us, first, set in (2. 7) $d = n = 1$ and $b = k = 0$. These variables were used until now in investigations of the nonlinear transonic equation [1]. This leads to the known conclusion that the perturbation propagation is defined by parabola [1]

$$x = f(y,t) = \frac{\omega}{2}(\varkappa + 1)\frac{\delta}{\varepsilon}t + \frac{1}{2}y^2 t^{-1}$$

which means that the perturbation propagates downstream at infinite velocity, and in addition t = const is a characteristic for (2. 2) [1]. These two features represent a considerable shortcoming of Eqs. (2. 2) and (2. 5) represented in terms of variables x, y, z, t. It can be readily shown that then function $x = f(y, t)$ is the first term of expansion in terms of the small parameter δ / ε of the function that determines the exact perturbation boundary, which in this case is a circle.

We require the curve (2. 7) to be a circle for $\omega = 0$, which is reasonable, since for $\omega = 0$ or $\delta / \varepsilon = 0$ we have, in accordance with (2. 1), the exact solution of the exact equation (1. 1) $\Phi = x$. This requirement leads to the condition that $n = -2k$ when $b = 0$, and $n = 0$ if $b = -2d$. The equation of the line of perturbation originating at point $x = y = 0$ for $t = 0$ is of the form $(x - t)^2 + y^2 = t^2$, which is an expanding circle carried downstream by the sonic flow.

If in (2. 7) $\omega \neq 0$, we have an ellipse that differs slightly (by a quantity of order δ / ε) from that circle. For a three-dimensional flow the perturbation boundary is defined by surface $\xi = A\eta + B\eta^{-1}(y^2 + z^2)$. If $\omega = 0$, that surface is a sphere $(x - t)^2 + y^2 + z^2 = t^2$. Thus it is necessary to set in (2. 1) $\xi = dx$, $\eta = k(x - 2t)$ or $\eta = kx$, $\xi = d(x - 2t)$. In that case t = const is no longer a characteristic, the latter being defined by x = const or $x - 2t$ = const. Note that in the one-dimensional nonlinear theory both these shortcomings are absent, hence it is possible to use any variables.

Let us assume that in Eq. (2. 2) coefficients $C = (\varkappa + 1)d^2(b + d)$ and $2B = bn + dn + bk$ do not vanish. We pass in (2. 2) and (2. 5) to the new variables $\theta = \eta / (2B)$, $\Psi = C\psi$, $\Psi^* = C\psi^*$. The solution that defines the flow in the parturbation region (and satisfies the conditions of continuity of velocity components at transition through the perturbation boundary) can be written as

$$\Psi^* = \frac{\Omega}{2}\xi + \frac{\xi^2}{2\theta} - \frac{y^2\xi}{2\theta^2} + \frac{\Omega}{4}\frac{y^2}{\theta} + \frac{y^4}{8\theta^3}, \quad \Omega = C\omega$$

Finally, let us obtain approximate conditions at the inpenetrable surface. Specifying the latter in the form

$$y = y_0 + \delta^{3/2}\varepsilon^{-1/2}\sum_{k=0}^{\infty} f_k(\xi^\circ, \eta^\circ, z^\circ)\left(\frac{\delta}{\varepsilon}\right)^k, \quad y^\circ = y_0^\circ + \left(\frac{\delta}{\varepsilon}\right)^2 \sum_{k=0}^{\infty}\left(\frac{\delta}{\varepsilon}\right)^k f_k \quad (2.8)$$

where y_0 = const, and substituting (2. 1) and (2. 8) into the exact condition of impenetrability

$$\Phi_y - \Phi_x\frac{\partial y}{\partial x} - \Phi_z\frac{\partial y}{\partial z} = \frac{\partial y}{\partial t}$$

in the first approximation we obtain

$$(b + d)\partial f_0(\xi, \eta, z) / \partial\xi = \psi_y(y_0, \xi, \eta, z), \; b \neq -d \quad (2.9)$$

where the superscript is omitted.

Similar results are valid for small perturbations in a quiescent gas. Seeking the solution of Eqs. (1. 1) and (2. 4), expressed in terms of dimensionless variables (instead of P_*, ρ_* and a_* we use parameters P_0, ρ_0 and $a_0 = \sqrt{(\varkappa + 1) / 2}a_*$ of the quiescent gas), in the form (2. 1) and (2. 3) and rejecting in (2. 1) the term x in the expression for Φ , in the first approximation we obtain

$$b = \pm d, \quad 2(bn - dk)\psi_{\xi\eta} + (\varkappa + 1)bd^2\psi_\xi\psi_{\xi\xi} = \psi_{yy} + \psi_{zz} \qquad (2.10)$$

$$2(bn - dk)\frac{\partial\xi_0}{\partial\eta} + \left(\frac{\partial\xi_0}{\partial y}\right)^2 + \left(\frac{\partial\xi_0}{\partial z}\right)^2 = \frac{\varkappa+1}{2}bd^2(\psi_\xi + \psi_\xi^*), \quad \psi = \psi^*$$

For conditions at the shock front we use functions for $\xi = \xi_0$. The condition of impenetrability at the surface (2.8) for $y = y_0$ $(b \neq 0)$ is $b\partial f_0 / \partial\xi = \psi_y$. By specifying for the perturbation front originating in a quiescent gas $(\psi_\xi = \psi_\xi^* = 0)$ the form of circle $x^2 + y^2 = t^2$ (or of sphere $x^2 + y^2 + z^2 = t^2$), we find that $n = \mp k$, hence $\xi = d(x \pm t)$ and $\eta = k(x \mp t)$.

Thus for deriving the nonlinear equations for small unstable two- and three-dimensional perturbations of a sonic stream or in a quiescent gas it is necessary to use the characteristic variables of the related linear equations of one-dimensional flows. Although equations in terms of other variables can evidently be used, care must be taken to interpret these correctly. In particular, they can be used for defining flows whose unsteadiness becomes apparent only in the second approximation. Note that all solutions of the transonic equation in variables x and t [1] can be rewritten for Eqs. (2.2) and (2.10), by reducing these beforehand to the form appearing in [1]. This applies also to transformations that do not alter the form of the transonic equation (e.g. of that appearing in [4]) as well as the form of conditions at the shock front (or at a characteristic). Finally, a theorem of uniqueness, similar to that in [4] can be formulated for these equations.

REFERENCES

1. Ryzhov, O. S., Investigation of Transonic Flows in Laval Nozzles. Izd. VTs Akad. Nauk SSSR, Moscow, 1965.
2. Coull, J. D., Perturbation Methods in Applied Mathematics. Blaisdell, Waltham, Mass., USA, 1968.
3. Van Dyke, M., Perturbation Methods in Fluid Mechanics. Academic Press, N. Y. and London, 1964.
4. Mamontov, E. V., On the theory of unstable transonic flows. Dokl. Akad. Nauk SSSR, Vol. 185, № 3, 1969.

Translated by J. J. D.

UDC 539.385

AXISYMMETRIC CONTACT PROBLEM ON THE IMPRESSION OF AN ELASTIC CYLINDER INTO AN ELASTIC LAYER

PMM Vol. 40, № 1, 1976, pp. 79-84

A. N. ZLATIN and Ia. S. UFLIAND
(Leningrad)

(Received June 9, 1974)

A generalization is given of the problem on the impression of a circular stamp when the elastic stamp makes contact with an unbounded elastic layer. Application of the Hankel integral transform in the region of the layer and the properties of generalized orthogonality of eigenfunctions in the region of the circular cylinder (stamp) permits reducing the problem to an infinite system of